JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 2, May 2019
http://dx.doi.org/10.14403/jcms.2019.32.2.245

NOTES ON γ-OPEN SETS DEFINED BY γ-OPERATION ON A SUPRATOPOLOGICAL SPACE

Young Key Kim* and Won Keun Min**

Abstract. In this paper, the notion of γ-operation on a supratopological space is introduced. We found that the γ-operation induces a supratopology (topology) containing a given supratopology. We also introduce the notions of (γ, S)-continuous function and almost Γ-supracompact defined by γ-operation on a supratopological space and investigate some properties for such notions.

1. Introduction and Preliminaries

Let X be a non-empty set with the power set $expX$. A function $\gamma : expX \to expX$ is said to be *monotonic* [1] iff $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$. The monotonic function γ is called an *operation*. If γ is an operation, then a set $A \subseteq X$ is said to be γ-*open* [1] if $A \subseteq \gamma A$. For $A \subseteq X$, we denote by $i_\gamma A$ the union of all γ-open sets contained in A, i.e. the largest γ-open set contained in A. The complement of a γ-open set is said to be γ-*closed*. Any intersection of γ-closed sets is γ-closed, and for $A \subseteq X$, we denote by $c_\gamma A$ the intersection of all γ-closed sets containing A, i.e. the smallest γ-closed set containing A. Let γ and γ' be operations, respectively. Then a function $f : X \to Y$ is said to be (γ, γ')-*continuous* [3] if for each γ'-open set V in Y, $f^{-1}(V)$ is γ-open in X.

We recall the notion of γ-operation introduced in [2]: Let (X, τ) be a topological space, and $\gamma : expX \to expX$ a mapping such that

(1) $A \subseteq B \Rightarrow \gamma A \subseteq \gamma B$.

(2) $\gamma\emptyset = \emptyset$, $\gamma X = X$.

Received April 30, 2019; Accepted March 01, 2019.

2010 Mathematics Subject Classification: 54A05, 54B10, 54C10, 54D30.

Key words and phrases: (γ, S)-continuous, Γ-supraclosed graph, strongly Γ-supraclosed graph, γ-supracompact, almost Γ-supracompact.

This study was supported by 2017 Research Grant from Kangwon National University(No. 520170500).

**Corresponding author:wkmin@kangwon.ac.kr.

(3) For $A \subseteq X$ and an open set $G \subseteq X$, $G \cap \gamma A \subseteq \gamma(G \cap A)$.

The mapping γ is called an *associated operation* with μ on X.

In this paper, we introduce the notion of an *associated operation* with a supratopology S on any given nonempty set. We found that the γ-operation induces a supratopology (topology) containing a given supratopology. (See Theorem 2.5). We also study the notions of (γ, S)-continuous function, Γ-supraclosed graph, strongly Γ-supraclosed graph and almost Γ-supracompact defined by an associated γ_S-operation on a supratopological space.

THEOREM 1.1 ([1]). *Let γ be an operation and $A \subseteq X$. Then the statements are hold:*
(1) $i_\gamma A = X - c_\gamma(X - A)$; (2) $c_\gamma A = X - i_\gamma(X - A)$.

Let X be a nonempty set. A subclass $\mathcal{S} \subseteq expX$ is called a supratopology [4] on X if $\emptyset, X \in \mathcal{S}$ and $\mathcal{S}$ is closed under arbitrary union. $(X, \mathcal{S})$ is called a supratopological space. The members of $\mathcal{S}$ are called supraopen sets and a set is called supraclosed if the complement is a member of $\mathcal{S}$. For $A \subseteq X$, we denote by $Sint(A)$ the union of all supraopen sets contained in A, and by $Scl(A)$ the intersection of all γ-closed sets containing A.

2. Main Results

DEFINITION 2.1. Let $(X, \mathcal{S})$ be a supratopological space with a supratopology $\mathcal{S}$, and $\gamma : expX \to expX$ a mapping such that

(1) $A \subseteq B \Rightarrow \gamma A \subseteq \gamma B$.

(2) $\gamma\emptyset = \emptyset$, $\gamma X = X$.

(3) For $A \subseteq X$ and any supraopen set $G \subseteq X$, $G \cap \gamma A \subseteq \gamma(G \cap A)$.

We call the mapping γ an *associated operation* with a supratopology $\mathcal{S}$ on X. We will denote an associated operation γ with $\mathcal{S}$ by γ_s (simply γ).

THEOREM 2.2. *Let $(X, \mathcal{S})$ be a supratopological space and γ an associated operation with $\mathcal{S}$. Then every supraopen set is γ-open.*

Proof. Let G be supraopen in X. Then from (2) of Definition 2.1, $G = G \cap \gamma X \subseteq \gamma(G \cap X) \subseteq \gamma G$. Thus G is γ-open. □

THEOREM 2.3. *Let $(X, \mathcal{S})$ be a supratopological space and γ an associated operation with $\mathcal{S}$. Then the intersection of a supraopen set and a γ-open set is γ-open.*

Proof. Let G and H be a supraopen set and a γ-open set, respectively. Then $G \cap H \subseteq G \cap \gamma H \subseteq \gamma(G \cap H)$. Thus $G \cap H$ is γ-open. □

In general, the intersection of two γ-open sets is not γ-open as shown in the next example.

EXAMPLE 2.4. *For $X = \{a, b, c\}$, let $\mathcal{S} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ be a supratopology. Consider a mapping $\gamma : expX \to expX$ such as $\gamma(A) = Scl(Sint(A))$ for $A \subseteq X$. Then obviously, γ is an operation. Take two γ-open sets $A = \{a, b\}$ and $B = \{b, c\}$. For $A \cap B = \{c\}$, since $\gamma(A \cap B) = Scl(Sint(A \cap B)) = Scl(Sint(\{c\})) = \emptyset$, $A \cap B$ is not γ-open.*

From the above facts, we have the next result:

THEOREM 2.5. *Let $(X, \mathcal{S})$ be a supratopological space and γ an associated operation with $\mathcal{S}$. Then*
(1) the set of all γ-open sets is a supratopology containing $\mathcal{S}$;
(2) if for $A \subseteq X$ and any γ-open set $G \subseteq X$, $G \cap \gamma A \subseteq \gamma(G \cap A)$, then the set of all γ-open sets is a topology.

Proof. (1) Let H_i be any γ-open set for each $i \in J$. Then from (1) of Definition 2.1, $H_i \subseteq \gamma(H_i) \subseteq \gamma(\cup H_i)$. Thus $\cup H_i$ is γ-open. By Theorem 2.2, then the set of all γ-open sets is a supratopology containing $\mathcal{S}$.
(2) Let A and B be γ-open sets. Then $A \cap B \subseteq A \cap \gamma B \subseteq \gamma(A \cap B)$. So, $A \cap B$ is a γ-open set. Finally, from (1), the set of all γ-open sets is a topology containing $\mathcal{S}$.

□

DEFINITION 2.6. Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. Then a function $f : (X, \mu) \to (Y, \nu)$ is (γ, S)-continuous if for every supraopen set F in Y, $f^{-1}(F)$ is γ-open in X.

THEOREM 2.7. *Let $f : (X, \mu) \to (Y, \nu)$ be a function on supratopological spaces and γ an associated operation with μ. Then f is (γ, S)-continuous iff for each $x \in X$ and each supraopen set V containing $f(x)$, there exists a γ-open set U containing x such that $f(U) \subseteq V$.*

Proof. Suppose that f is (γ, S)-continuous. Then for each $x \in X$ and each supraopen set V containing $f(x)$, $f^{-1}(V)$ is γ-open. Set $U = f^{-1}(V)$. Then the γ-open U satisfies that $x \in U$ and $f(U) \subseteq V$.
For the converse, let V be a supraopen set in Y. Then for each $x \in f^{-1}(V)$, there exists a γ-open set U_x such that $x \in U_x \subseteq f^{-1}(V)$. So $f^{-1}(V) = \cup U_x$ and by Theorem 2.5, it is γ-open. □

THEOREM 2.8. *Let $f : (X, \mu) \to (Y, \nu)$ be a function on supratopological spaces and γ an associated operation with μ. Then a function f is (γ, S)-continuous iff $f^{-1}(Sint(B)) \subseteq \gamma f^{-1}(B)$ for $B \subseteq Y$.*

Proof. Let f be (γ, S)-continuous and $B \subseteq Y$. Since $f^{-1}(Sint(B))$ is γ-open and γ is monotonic,

$$f^{-1}(Sint(B)) \subseteq \gamma f^{-1}(Sint(B)) \subseteq \gamma f^{-1}(B).$$

For the converse, let B be a supraopen set in Y. Then $f^{-1}(B) = f^{-1}(Sint(B)) \subseteq \gamma f^{-1}(B)$. So, $f^{-1}(B)$ is γ-open in X. □

THEOREM 2.9. *Let $f : (X, \mu) \to (Y, \nu)$ be a function on supratopological spaces and γ an associated operation with μ. Then the following are equivalent:*

(1) f is (γ, S)-continuous.
(2) $f^{-1}(Sint(B)) \subseteq i_\gamma f^{-1}(B)$ for $B \subseteq Y$.
(3) $c_\gamma f^{-1}(B) \subseteq f^{-1}(Scl(B))$ for $B \subseteq Y$.
(4) $f(c_\gamma A) \subseteq Scl(f(A))$ for $A \subseteq X$.

Proof. Straightforward. □

Let (X, μ) be a supratopological space and γ an associated operation with μ. Then X is called γ-T_2 (respectively, ST_2 [4]) if for every two distinct points x and y in X, there exist two γ-open sets (respectively, supraopen sets) U and V containing x and y, respectively, such that $U \cap V = \emptyset$.

DEFINITION 2.10. Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. A function $f : X \to Y$ has a *γ-supraclosed graph* (resp. *strongly γ-supraclosed graph*) if for each $(x, y) \in (X \times Y) - G(f)$, there exist a γ-open set U and a supraopen set V containing x and y, respectively, such that $(U \times V) \cap G(f) = \emptyset$ (resp. $(U \times Scl(V)) \cap G(f) = \emptyset$), where $G(f) = \{(x, f(x)) : x \in X\}$.

LEMMA 2.11. *Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. A function $f : X \to Y$ has a γ-supraclosed graph (resp. strongly γ-supraclosed graph) if for each $(x, y) \notin G(f)$, there exist a γ-open set U and a supraopen set V containing x and y, respectively, such that $f(U) \cap V = \emptyset$ (resp. $f(U) \cap Scl(V) = \emptyset$).*

Proof. Obvious. □

THEOREM 2.12. *Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. If $f : X \to Y$ is (γ, S)-continuous and Y is a ST_2 space, then f has a strongly γ-supraclosed graph.*

Proof. Let $x, y \in (X \times Y) - G(f)$. Then $y \neq f(x)$, and there exist two supraopen sets U and V such that $f(x) \in U$, $y \in V$ and $U \cap V = \emptyset$. It implies that $U \cap Scl(V) = \emptyset$. Since f is (γ, S)-continuous, by Theorem 2.7, there exists a γ-open set W of x such that $f(W) \subseteq U$. So $f(W) \cap Scl(V) = \emptyset$. Thus by Lemma 2.11, f has a strongly γ-supraclosed graph. □

THEOREM 2.13. *Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. If $f : X \to Y$ is a surjective function with a strongly γ-supraclosed graph, then Y is ST_2-space.*

Proof. Let y_1 and y_2 be distinct points in Y. Then there exists $x \in X$ such that $f(x) = y_1$. Since $(x, y_2) \notin G(f)$ and f has a strongly γ-supraclosed graph, there exist a γ-open set U and a supraopen set V of x and y_2, respectively, such that $f(U) \cap Scl(V) = \emptyset$. So, $y_1 \notin Scl(V)$. Now, there exists a supraopen set G of y_1 such that $G \cap V = \emptyset$. Hence, Y is ST_2. □

THEOREM 2.14. *Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. If $f : X \to Y$ is (γ, S)-continuous injection with a γ-supraclosed graph, then X is γ-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X. Then $f(x_1) \neq f(x_2)$ and $(x_1, f(x_2)) \in (X \times Y) - G(f)$. By hypothesis, there exist a γ-open set U and a supraopen set V of x_1 and $f(x_2)$, respectively, such that $(U \times V) \cap G(f) = \emptyset$. Since f is (γ, S)-continuous, there exists a γ-open set H containing x_2 such that $f(H) \subseteq V$. It implies that $f(H) \cap f(U) = \emptyset$. So, $H \cap U = \emptyset$ and X is γ-T_2. □

Let (X, μ) be a supratopological space and γ an associated operation with μ. A collection $\mathbf{S} = \{S_i \subseteq X : S_i \text{ is } \gamma\text{-open}, i \in I\}$ is called a *γ-open cover* for X if $X = \cup_{i \in I} S_i$. The space (X, μ) is said to be *γ-supracompact* (resp., *almost Γ-supracompact*) if for each γ-open cover $\mathbf{S} = \{S_i \subseteq X : S_i \text{ is } \gamma\text{-open}, i \in I\}$, there exists a finite index set $F \subseteq I$ such that $X = \cup_{i \in F} S_i$ (resp., $X = \cup_{i \in F} c_\gamma(S_i)$).

And we recall that a supratopological space (X, μ) is said to be *almost supracompact* if for each supraopen cover $\mathbf{C} = \{G_i \subseteq X : G_i \text{ is supraopen}, i \in I\}$, there exists a finite index set $F \subseteq I$ such that $X = \cup_{i \in F} Scl(G_i)$.

THEOREM 2.15. *Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. Let $f : X \to Y$ be a surjective (γ, S)-continuous function. If X is γ-supracompact, then Y is supracompact.*

Proof. Obvious. □

THEOREM 2.16. *Let (X, μ) and (Y, ν) be supratopological spaces and γ an associated operation with μ. Let $f : X \to Y$ be a (γ, S)-continuous and surjective function. If X is almost Γ-supracompact, then Y is almost supracompact.*

Proof. Let $\mathcal{S} = \{S_i : i \in J\}$ be a supraopen cover of Y. Then $\{f^{-1}(S_i) : S_i \in \mathcal{S}, i \in J\}$ is a γ-open cover of X and by almost Γ-supracompactness of X, there is a finite collection
$\{c_\gamma f^{-1}(S_{j_1}), c_\gamma f^{-1}(S_{j_2}), \cdots, c_\gamma f^{-1}(S_{j_n}) : S_j \in \mathcal{S}, j = j_1, j_2, \cdots, j_n\}$ such that $X \subseteq \cup c_\gamma f^{-1}(S_j)$. Then from Theorem 2.9,

$$Y = f(X) \subseteq f(\cup c_\gamma f^{-1}(S_j)) \subseteq \cup f(f^{-1}(Scl(S_j))) \subseteq \cup Scl(S_j).$$

Hence Y is almost supracompact. □

3. Acknowledgments

The authors are thankful to the referee for his/her useful suggestions.

References

[1] A. Császár, *Generalized Open Sets,* Acta. Math. Hungar. **75** (1997), no. 1-2, 65-87.
[2] A. Császár, *γ-compact Spaces,* Acta. Math. Hungar. **87** (2000), no. 1-2, 99-107.
[3] A. Császár, *γ-connected Sets,* Acta. Math. Hungar. **101** (2003), no. 4, 273-279.
[4] A. S. Mashhourr, A. A. Allam, F. S. Mahmoud and F. H. Khadr, *On Supratopological Spaces,* Indian J. Pure Appl. Math. **14**(4) (1983), 502-510.

*
Department of Mathematics
MyongJi University
Yongin 17058, Republic of Korea
E-mail: ykkim@mju.ac.kr

**
Department of Mathematics
Kangwon National University
Chuncheon 24341, Republic of Korea
E-mail: wkmin@kangwon.ac.kr